Knot Quandles

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Abstract

Is there an easy way to tell if two knots are the same or different? We will explore the idea of quandles and connect them to knots. We will discuss some common quandles, including the Alexander quandle.

Contents

1 Introduction

The idea of a quandle was first introduced in 1943 by Mituhisa Takasaki, when he was looking for an algebraic structure to represent reflections. He called it a Kei, and importantly, it did not satisfy the property of associativity [10]. The idea was revisited nearly 40 years later by David Joyce who related them to knots [5]. As a fun fact, when Joyce was coming up with the word to describe the structure, he had distributive algebra (which he thought was too long), piffle, trindle, and quagle, before finally settling on 'quandle' [1].

2 What are Quandles?

Definition 1. *A quandle is a set X along with a binary operations $\triangleright$ that satisfies the following axioms:*

1. $a \triangleright a = a$ *for all* $a \in X$.
2. *For all pairs* $b, c \in X$, *there is a unique* $a \in X$ *such that* $a \triangleright b = c$
3. $(a \triangleright b) \triangleright c = (a \triangleright c) \triangleright (b \triangleright c)$ *for all* $a, b, c \in X$.

The first axiom is called *idempotence*. The second axiom is often called *right-invertibility*. This is because if you fix some $b \in X$ and know the result $a \triangleright b$, then Axiom 2 states you can uniquely determine what a is. In other words, we can define a new inverse operation $\triangleright^{-1}$ that satisfies $(a \triangleright b) \triangleright^{-1} b = a$ for all $a, b \in X$ (essentially allowing one to 'go back' and get the unique solution). We can think of $c = a \triangleright b$ as the same as $a = c \triangleright^{-1} b$. Thus we also want $(a \triangleright^{-1} b) \triangleright a = a$ (we should be able to 'go back' in both directions). This second operation will ease notation in later sections. The third axiom states that the operation is right self-distributive.

As a note, this operation is often not commutative nor associative.

We will now show some examples of common quandles. The following is called the *dihedral quandle*.

Example 2. *Let X be the integers mod n with operation $a \triangleright b = 2b - a \pmod{n}$. We will show X is a quandle under $\triangleright$.*

Proof. We will show the three axioms hold.

(Axiom 1): Let $a \in X$. Then $a \triangleright a = 2a - a \pmod{n} = a \pmod{n}$ as desired.

(Axiom 2): Let $b, c \in X$. Then choosing $a = 2b - c \pmod{n} \in X$ satisfies $a \triangleright b = 2b - (2b - c) \pmod{n} = c \pmod{n}$, and equality will not hold for any $a \neq 2b - c \pmod{n}$, which is easy to check.

(Axiom 3): Let $a, b, c \in X$. Then observe

$$\begin{aligned}(a \triangleright b) \triangleright c &= (2b - a) \triangleright c = 2c - (2b - a) \pmod n \\ &= 2c - 2b + a \pmod n \\ &\text{and} \\ (a \triangleright c) \triangleright (b \triangleright c) &= (2c - a) \triangleright (2c - b) \\ &= 2(2c - b) - (2c - a) \pmod n \\ &= 2c - 2b + a \pmod n \\ &= a \triangleright (b \triangleright c)\end{aligned}$$

□

We can generalize the dihedral quandle to the *Takasaki quandle*.

Example 3. *Let G be an abelian group with operation $a \triangleright b = 2b - a$ (using the additive operation of G). Then G is a quandle under $\triangleright$, called the Takasaki quandle.*

The proof is similar to the previous proof, as this quandle is simply a generalization.

Example 4. *Let G be a group. We will show G is a quandle under the operation $a \triangleright b = b^{-1}ab$, called the conjugation quandle.*

Proof. We will show the three axioms hold.

(Axiom 1): Let $g \in G$. Then $g \triangleright g = g^{-1}gg = eg = g$ as desired.

(Axiom 2): Let $b, c \in G$. Since the group operation is a binary operation, there is exactly one element $a = bcb^{-1} \in G$. Now note that $a \triangleright b = b^{-1}(bcb^{-1})b = c$ as desired.

(Axiom 3): Let $a, b, c \in G$. Then observe that

$$\begin{aligned}(a \triangleright b) \triangleright c &= (b^{-1}ab) \triangleright c = c^{-1}(b^{-1}ab)c \\ &\text{and} \\ (a \triangleright c) \triangleright (b \triangleright c) &= (c^{-1}ac) \triangleright (c^{-1}bc) = (c^{-1}bc)^{-1}(c^{-1}ac)(c^{-1}bc) \\ &= c^{-1}b^{-1}cc^{-1}acc^{-1}bc = c^{-1}b^{-1}abc \\ &= (a \triangleright b) \triangleright c\end{aligned}$$

as desired. So G is a quandle under the operation $\triangleright$. □

3 Knots and Quandles

Our goal is give some visual intuition on the relation between oriented knots and quandles. Given some oriented knot, at any crossing, there are two possible cases - either we have a left handed crossing or right handed crossing.

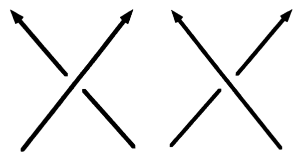

Figure 1: Right-handed and left-handed crossing, respectively

Let the set X be the oriented strands of the knot and label each strand a letter to keep track of them (I will refer to each strand by the letter). We then can think of the quandle operation of $\triangleright$ as meaning "passing under a right handed crossing," and $\triangleright^{-1}$ as meaning "passing under a left handed crossing". That is, we can label Figure 1 according to these rules and get Figure 2.

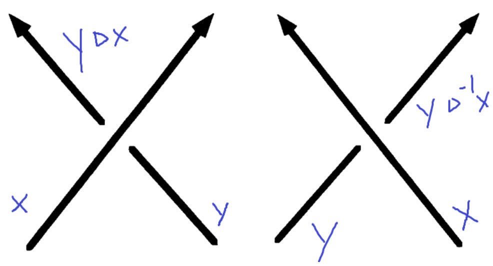

Figure 2: Right-handed and left-handed crossing, respectively

Let's now show that this interpretation is equivalent to the Reidemeister moves. Suppose that a knot was labelled according to these rules. In Figure 3, we see that when applying Reidemeister Move 1, we must have $a \triangleright a = a$ because on one hand, when passing under the right hand crossing, we label this strand $a \triangleright a$. On the other hand, we know that this strand is equal to a, so we satisfy idempotence. Additionally, we see that when dealing with a left handed crossing, we also have $a \triangleright^{-1} a = a$, for the same reason.

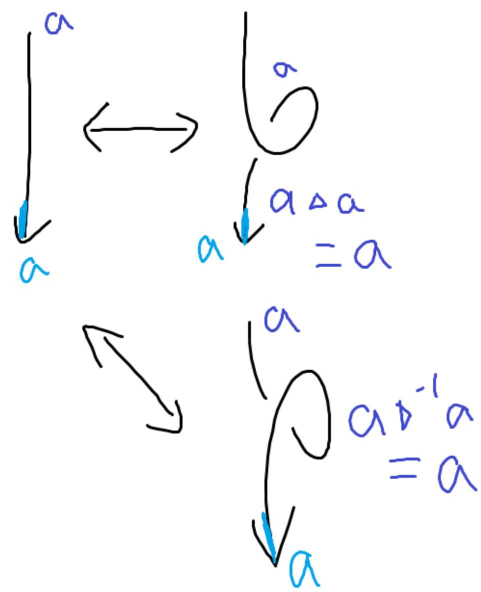

Figure 3: Reidemeister Move 1

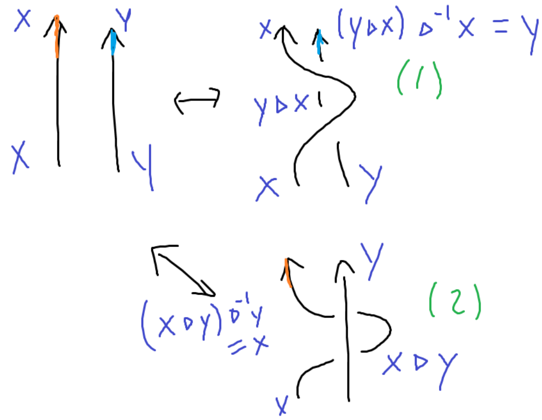

Figure 4: Reidemeister Move 2

For Reidemeister Move 2 (Figure 4), in (1), we see that on one hand, the light blue strand is $(y \triangleright x) \triangleright^{-1} x$ (y passes under a right crossing of x, then this whole strand passes under a left crossing of x), which must be equal to y if we undid the Reidemeister move. So $(y \triangleright x) \triangleright^{-1} x = y$ which satisfies the right-invertibility rule. We see a similar behavior in (2), which one can verify.

Reidemeister Move 3 gives the self-distributive property of quandles. From Figure 5, we have that the orange strands are equal, that is, $(x \triangleright z) \triangleright (y \triangleright z) = (x \triangleright y) \triangleright z$, which is exactly Axiom 3. Note that while the other configurations of Move 3 were not shown, they will all lead to the self-distributive property (some with a greater degree of difficulty).

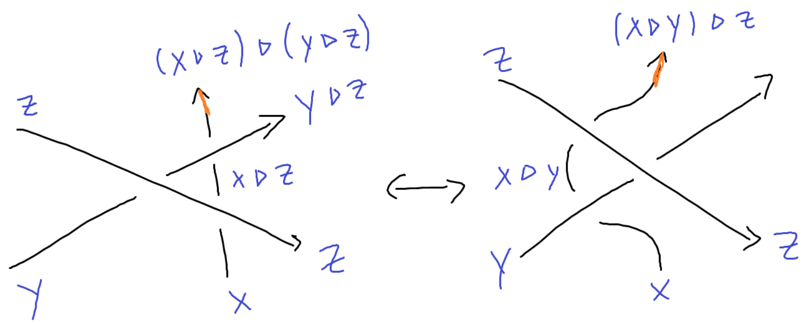

Figure 5: Reidemeister Move 3

As seen from the previous section, when applying the Reidemeister moves, there is exactly one way to label the new diagram. In other words, applying a Reidemeister move cannot introduce any ambiguity on how to label the diagram. Thus, the total number of ways to label a diagram under some quandle operation is an invariant, called the *counting invariant* [2].

As an example, consider the set $\mathbb{Z}_3$ and corresponding quandle operation $x \triangleright y = 2x + 2y \pmod 3$. The trefoil knot has nine possible labellings with $\mathbb{Z}_3$, as seen below. One can easily check that the quandle operation is satisfied for every strand.

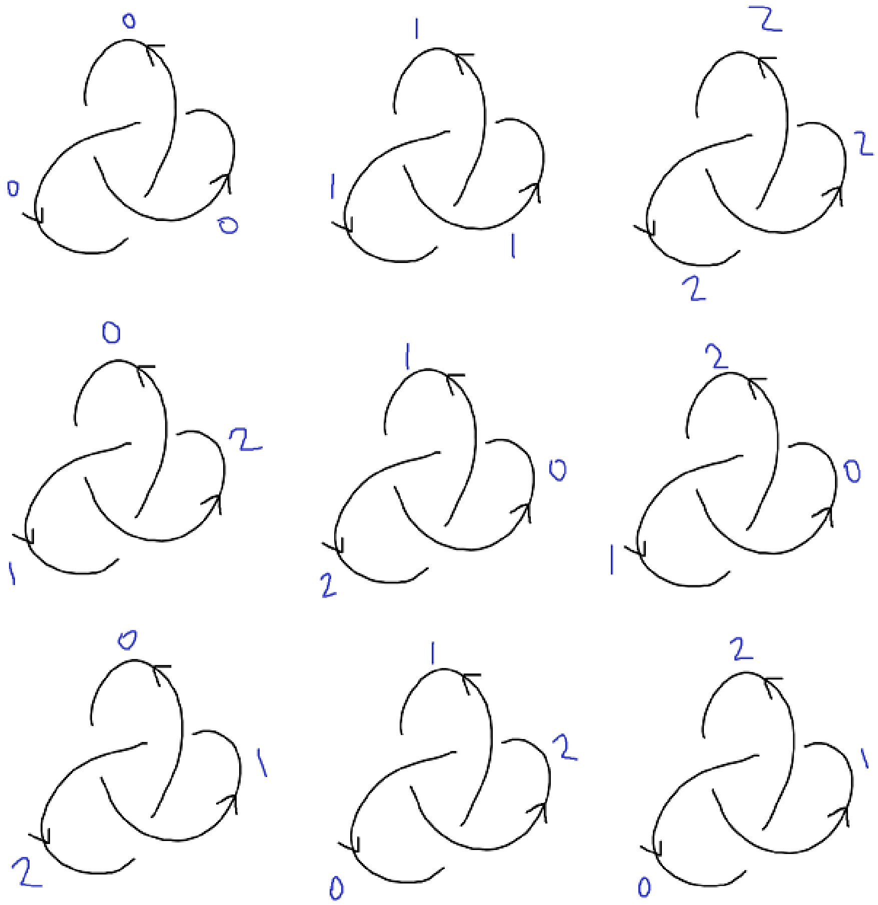

Figure 6: 9 Labelings for the Trefoil

4 The Alexander Quandle

Before we discuss the Alexander quandle, we must define what a module is. One can think of a module as a generalization of a vector space.

Definition 5. *Let R be a commutative ring with identity. A module M is an abelian group M with an operation $+$, together with an operation of multiplication $\cdot : R \times M \to M$ such that for all $r, s \in R$ and $x, y \in M$ we have*

1. $r \cdot (x + y) = r \cdot x + r \cdot y$
2. $(r + s) \cdot x = r \cdot x + s \cdot x$
3. $(rs) \cdot x = r \cdot (s \cdot x)$
4. $1 \cdot x = x$

If the above holds, we will say that M is an R-module, or that M is a module over R. The operation $\cdot$ is called scalar multiplication, and is often omitted.

Example 6. *A vector field over a field $\mathbb{F}$ is an $\mathbb{F}$-module.*

The multiplication operation $c \cdot \vec{v}$ is multiplying each component of $\vec{v}$ by the constant c. The four axioms directly fall from the properties of vector fields.

Example 7. *Let M be any abelian group under $+$, and let $R = \mathbb{Z}$ (the scalars are integers). We will now define the scalar operation. Let $r \in R$ and $a \in M$.*

1. *If $r > 0$, then we define $r \cdot a = \underbrace{a + \ldots + a}_{r \text{ times}}$.*
2. *If $r = 0$, then we define $r \cdot a = 0$.*
3. *If $r < 0$, then we define $r \cdot a = \underbrace{-a - \ldots - a}_{|r| \text{ times}}$.*

We will show that M is a R-module, thus showing that every abelian group is a $\mathbb{Z}$-module.

Proof. Let $r, s \in \mathbb{Z}$ and $x, y \in M$.

(1): If $r > 0$,

$$\begin{aligned} r \cdot (x+y) &= \underbrace{(x+y) + \ldots + (x+y)}_{r \text{ times}} \\ &= \underbrace{x + \ldots + x}_{r\text{times}} + \underbrace{y + \ldots + y}_{r \text{ times}} \\ &= r \cdot x + r \cdot y \end{aligned}$$

The case when $r = 0$ and $r < 0$ can be easily checked.

(2): We must check the cases when $r+s$ is positive, when $r+s$ is negative, and when it is 0. We will show the case when $r+s$ is positive and $r, s > 0$. The other cases are not too hard to show, but require more casework to deal with the signs of r and s.

$$\begin{aligned} (r+s) \cdot x &= \underbrace{x + \ldots + x}_{r+s \text{ times}} \\ &= \underbrace{x + \ldots + x}_{r \text{ times}} + \underbrace{x + \ldots + x}_{s \text{ times}} \\ &= r \cdot x + s \cdot x \end{aligned}$$

(3): If r or s is 0, it is easy to show that the products on both sides will be 0. So assume that they are nonzero. We have 4 cases to check (the signs of r and s), and we will show the case where $r > 0$ and $s < 0$, though the other cases are similar.

$$\begin{aligned} (rs) \cdot x &= \underbrace{-x - \ldots - x}_{|rs| \text{ times}} \\ &= \underbrace{(\underbrace{-x - \ldots - x}_{|s| \text{ times}}) + \ldots + (\underbrace{-x - \ldots - x}_{|s| \text{ times}})}_{r \text{ times}} \\ &= r \cdot (\underbrace{-x - \ldots - x}_{|s| \text{ times}}) \\ &= r \cdot (s \cdot x) \end{aligned}$$

(4): We have $1 \cdot x = \underbrace{x}_{1 \text{ times}} = x$. ∎

While there is much more to discuss about modules, we are now ready to define the *Alexander quandle.*

Definition 8. *Let $R = \mathbb{Z}[t, t^{-1}]$ be a ring, and let M be a module over the ring R. Then the Alexander quandle A_t is the set M with quandle operation $x \triangleright y = tx + (1-t)y$. The inverse quandle operation is $x \triangleright^{-1} y = t^{-1}x + (1-t^{-1})y$.*

$\mathbb{Z}[t, t^{-1}]$ is a multivariate polynomial ring with integer coefficients. A typical element in R has the form $a_n t^n + \ldots + a_1 t + a_0 - a_{-1}t^{-1} + \ldots + a_{-m}t^{-m}$ where the coefficients $a_n, \ldots, a_0, \ldots, a_{-m} \in \mathbb{Z}$.

Note that putting $t = -1$ gives us the quandle operation $x \triangleright y = 2y - x$, which is exactly the Takasaki quandle.

Let us show that A_t is indeed a quandle under this operation.

Proof. Let $a, b, c \in M$.

(Axiom 1): Note that $a \triangleright a = ta + (1-t)a = ta + a - ta = a$.

(Axiom 2): Firstly,

$$\begin{aligned}(a \triangleright b) \triangleright^{-1} b &= (ta + (1-t)b) \triangleright^{-1} b \\ &= t^{-1}(ta + b - bt) + (1 - t^{-1}b \\ &= a + bt^{-1} - b + b - bt^{-1} \\ &= a\end{aligned}$$

Also,

$$\begin{aligned}(a \triangleright^{-1} b) \triangleright b &= (t^{-1}a + (1-t^{-1})b) \triangleright b \\ &= t(t^{-1}a + b - bt^{-1}) + (1-t)b \\ &= a + bt - b + b - bt \\ &= a\end{aligned}$$

(Axiom 3): Lastly,

$$\begin{aligned}(a \triangleright b) \triangleright c) &= (ta + (1-t)b) \triangleright c \\ &= t(ta + b - bt) + (1-t)c \\ &= at^2 + bt - bt^2 + c - ct\end{aligned}$$

Also,

$$
\begin{aligned}
(a \triangleright c) \triangleright (b \triangleright c) &= (ta + (1-t)c) \triangleright (tb + (1-t)c) \\
&= t(at + c - ct) + (1-t)(bt + c - ct) \\
&= at^2 + ct - ct^2 + bt + c - ct - bt^2 - ct + ct^2 \\
&= at^2 + bt - bt^2 + c - ct
\end{aligned}
$$

as desired. □

5 The Trefoil and the Alexander Polynomial

The Alexander polynomial can be defined in many ways, but we will show the relationship between the Alexander quandle and Alexander polynomial specifically for the trefoil. Now consider the following labelling from some module M (over $R = \mathbb{Z}[t, t^{-1}]$) of the the trefoil (let $a, b \in M$).

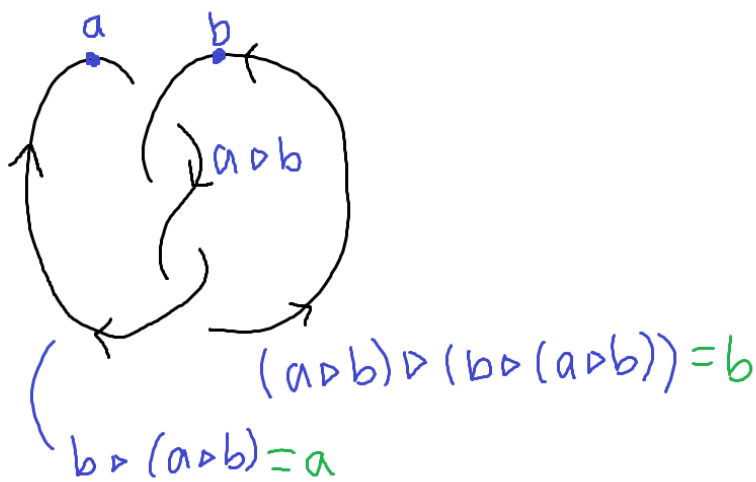

Figure 7: Quandle labelling of trefoil

We see that after labelling, we must have the relations $b \triangleright (a \triangleright b) = a$ and $(a \triangleright b) \triangleright (b \triangleright (a \triangleright b)) = b$ since these arcs are the same arc, respectively. Now using the Alexander quandle operation $a \triangleright b = at + (1-t)b$, we have for the first equation,

$$\begin{aligned} b \triangleright (a \triangleright b) &= a \\ b \triangleright (at + (1-t)b) &= a \\ bt + at + b - bt - at^2 - bt + bt^2 - a &= 0 \\ b - bt + bt^2 - a + at - at^2 &= 0 \\ b(1 - t + t^2) - a(1 - t + t^2) &= 0 \\ (b - a)(1 - t + t^2) &= 0 \end{aligned}$$

For the second equation (with intermediate simplification removed),

$$\begin{aligned}(a \triangleright b) \triangleright (b \triangleright (a \triangleright b)) &= b \\ (at + (1-t)b) \triangleright ((b-a)(1-t+t^2)) &= b \\ (t-1)(b-a)(1-t+t^2) &= 0\end{aligned}$$

If both of these equations must be true, we must have that $(b-a)(1-t+t^2) = 0$ surely. The $(b-a)$ means that there is always a trivial labelling of the trefoil by choosing $b = a$ (since by Axiom 1, $a \triangleright a = a$ for all arcs). We see the Alexander polynomial pop out with $1 - t + t^2$ (multiplied by t^{-1} to have symmetry), with $\Delta_{3_1}(t) = t^{-1} - 1 + t$

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